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1. Determine all pairs of non-negative integers (m, n) which are solutions to the equation $m^3 + n^3 = m^2 n + mn^2 + 2013$.

Solution. Equality $m^3 + n^3 = m^2 n + mn^2 + 2013$ one can rewrite as $(m+n)(m-n)^2 = 2013$. Since 2013=1·3·11·61, then $(m-n)^2 = 1$ and $m+n=2013$,i.e.,

$m=1007$; $n=1006$ and $m=1006$; $n=1007$.

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2. Let two functions $f(x)$ and $g(x)$ satisfy the following conditions

$$f(2x+1)+g(3-x)=x$$

and

$$f\left(\frac{3x+5}{x+1}\right)+2g\left(\frac{2x+1}{x+1}\right)=\frac{x}{x+1}$$

for all real numbers $x \neq 1$. What is $f(2013)$?

Solution. From $2x+1 = 2013$, we have $x = 1006$ and $3 - x = -1003$.

From $\frac{3x+5}{x+1} = 2013$, we have $x = -\frac{2008}{2010}$ and $\frac{2x+1}{x+1} = -1003$.

From first equation we have $f(2013) + g(-1003) = 1006$

and from second $f(2013) + 2g(-1003) = -1004$.

Solving the system of equations

$$f(2013) + g(-1003) = 1006$$

$$f(2013) + 2g(-1003) = -1004,$$

we have $f(2013) = 3016$.

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3. Let ΔABC be a triangle with $|AB|= 4$, $|AC|=3$ and $\angle BAC = 90^\circ$. Let E and D be points on sides $[AB]$ and $[BC]$ respectively, such that $\angle EDC = 90^\circ$. If the area of the triangle ΔBDE is equal to area of quadrangle $EDCA$, then what is $|DC|$?

Solution. Let $|DC|$=x, then |BD|=5-x. It is evident that $\frac{|ED|}{|BD|} = \frac{|AC|}{|AB|}$, i.e., $\frac{|ED|}{5-x} = \frac{3}{4}$ and $|ED| = \frac{15-3x}{4}$. The area of the triangle ΔABC ie equal to 6 and $(5-x)\cdot\frac{15-3x}{4} = 6$. Solving this equation we have $x = 5 - 2\sqrt{2}$.

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4. Solve the equation

$$\sqrt[3]{6+x}+\sqrt[3]{6-x}=\sqrt[3]{3}$$

Solution. Raising both sides to the third power we have

$$6+x+3\sqrt[3]{(6+x)^2}\cdot\sqrt[3]{6-x}+3\sqrt[3]{6+x}\cdot\sqrt[3]{(6-x)^2}+6-x=3$$

or

$$6+x+6-x+3\sqrt[3]{6+x}\cdot\sqrt[3]{6-x}\left(\sqrt[3]{6+x}+\sqrt[3]{6-x}\right)=3$$

and simplifying we have

$$12+3\sqrt[3]{3(36-x^2)}=3 \quad \text{or} \quad \sqrt[3]{3(36-x^2)}=-3$$

and $x^2=45$ such that $x=\pm\sqrt{45}$.

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5. There is a positive integer *A*. Two operations are allowed: increasing this number by 3 and deleting a digit equal to 1 from any position.

(i) Is it possible to obtain 212 by applying these operations several times to *A*=211 ?

(ii) Is it possible to obtain 2013 by applying these operations several times to *A*=2012 ?

(iii) Is it always possible to obtain *A* + 1 by applying these operations several times to any *A*?

First Solution:

(i) 211→ 2 → 2+3·70=212;

(ii) 2012→202→202+3·3=211→21→21+3·664=2013;

(iii) From A applying first operation one can get (A+1)11, since (A+1)11-A is divisible by 3. Note that (A+1)11= (A+1)·100+11=100A+ 111, such that (A+1)11-A = 99·A+111, i.e., divisible by 3.

Now applying second operation from (A+1)11 one can get A+1.

Second Solution:

(i) 211+999=1210→210→210+999=1209→209→209+3=212;

(ii) 2012+9999=12011→2011→2011+9999=12010→2010→2010+3=2013.