## Solution for Multiple Choice Questions

1. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)\Rightarrow\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}=\frac{3}{2}$.

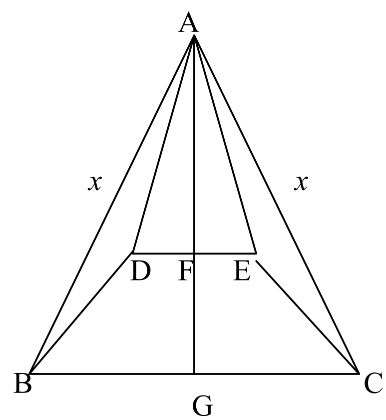
2. There are $n-3$ diagonals through each vertex and each diagonal is counted twice. So there are $\frac{n(n-3)}{2}$ diagonals.

3. $r=\frac{S}{p}$ where $p=\frac{a+b+c}{2}=\frac{90}{2}=45$. Since $9^2+40^2=41^2$, $\angle ACB=90^\circ$ and $S=\frac{9\times 40}{2}=180$. Then $r=\frac{180}{45}=4$.

4. From 1 to 9, there is one such integer. From $10n$ to $10n+9$, where $1\le n\le 199$, there are two such integers. From 2000 to 2007, there is one such integer. So the total is 400.

5. Since the last digit for $n!, n\ge 5$ is zero and the last digit of $1!+2!+3!+4!=1+2+6+24=33$ is 3, the answer is 3.

6. Let the angle bisector of intersect $DE$ at $F$ and $BC$ at $G$.. By symmetry, $AG$ is the perpendicular bisector to $DE$ and $BC$. Now $AF=\sqrt{7-\frac{1}{4}}=\frac{3\sqrt{3}}{2}$ . If $AB=x$, then $AG=\frac{x\sqrt{3}}{2}$.



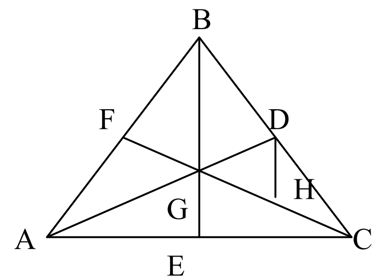

So

$$(x-3)\frac{\sqrt{3}}{2}=AG-AF=FG=\sqrt{2-\left(\frac{x-1}{2}\right)^2}$$

$$\Rightarrow x=\frac{5+\sqrt{13}}{2}.$$

7. The smallest 5 different prime divisors $2\cdot3\cdot5\cdot7\cdot11=2310>500$, there are no number less than 500 can have more than 4 different prime divisors.
The four prime divisor can be chosen only by $2\cdot3\cdot5\cdot7=210$, $2\cdot3\cdot5\cdot11=330$ and $2\cdot3\cdot7\cdot11=462$.

8. Let *G* be the centroid and *H* be the midpoint of *CG*. It maybe proved that *GD*=3, *DH*=4 and *HG*=5. Hence *DH* is a right-angled triangle with area 6. Now this triangle is 1/12 of Δ*ABC* hence area of Δ*AB*C= 72 square units.




9. Divide the carpet into $41^2$ square of size 3×3 and consider the central square of any of these 3×3 square. No matter whether it is a red or blue square, we have 4 red and 5 blue squares in 3×3 squares. Hence the total number of red square $=4\times41^2=6724$.

10. Let the sides of such triangle be *n*–1, *n*, *n*+1. Then $(n-1)+n+(n+1)\le100$ and $(n-1)+n>n+1$ imply $n>2$ and $n\le33$, that is $n=3,4,\ldots,33$ when $n=3$, $2^2+3^2<4^2$, the triangle is obtuse; when $n=4, 3^2+4^2=5^2$, the triangle is right; when $n\ge5, n^2+(n-1)^2-(n+1)^2=n^2-4n=n(n-4)>0$ the triangle is acute. Thus the required number is 33–4=29.

11. First $\frac{13}{7}<\frac{n+k}{n}=1+\frac{k}{n}<\frac{15}{8}$ gives $\frac{6}{7}<\frac{k}{n}<\frac{7}{8}$ or $\frac{48}{56}<\frac{k}{n}<\frac{49}{56}\Rightarrow\frac{96}{112}<\frac{k}{n}<\frac{98}{112}$. Hence $n=112,(k=97)$ is the answer.

12. Let *x* and *y* are number of rabbit and pheasant respectively. Then we have
$x+y=35$ (1)
$4x+2y=94$ (2)
(2)–2×(1) $\Rightarrow$ $2x=24\Rightarrow x=12$. Then $y=23$, the difference is 11.

13. $$\frac{1}{a+\sqrt{a^2+1}}=\frac{\left(\sqrt{a^2+1}-a\right)}{\left(a+\sqrt{a^2+1}\right)\left(\sqrt{a^2+1}-a\right)}=\sqrt{a^2+1}-a.$$

14. $x^{17}$ can only be obtained by multiplying two $x^5$ and one $x^7$. There are 20 ways to get $x^7$ and $C_2^{19}=171$ ways to get two $x^5$ in the remaining 19 factors. So the answer is $20\times171=3420$.

15. Let $m$ and $n$ be integral roots of $x^2+ax+8a=0$, with $m\le n$. Then $x^2+ax+8a=(x-m)(x-n)=x^2-(m+n)x+mn$. We have $a=-(m+n)$ and $8a=mn$. Thus $a$ is an integer and $-8(m+n)=mn$, get

$mn + 8m + 8n = 0$ or $(m+8)(n+8) = 64$. Consider the factorization of 64,
$64\times 1 = 32\times 2 = 16\times 4 = 8\times 8 = 4\times 16 = 2\times 32 = 1\times 64 =$
$(-64)\times(-1) = (-32)\times(-2) = (-16)\times(-4) = (-8)\times(-8) = (-4)\times(-16) =$
$(-2)\times(-32) = (-1)\times(-64)$ we obtain 8 distinct pairs of $m$ and $n$.

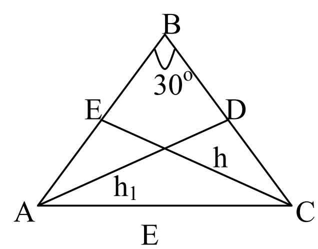
16. From $x^2 + xy + y^2 = 0$, get $\frac{xy}{(x+y)(x+y)} = 1$. Let $t = \frac{x}{x+y}$, then $\frac{1}{t} = \frac{y}{x+y}$ and $t + \frac{1}{t} = 1$, $t^2 - t + 1 = 0 = (t+1)(t^2 - t + 1) = t^3 + 1 \Rightarrow t^3 = -1$.

Thus $t^{2007} = (t^3)^{669} = -1$ and hence $t^{2007} + \frac{1}{t^{2007}} = -2$.

17. The equation can be rewritten as $(x^3 - b)^2 = 100$, which implies $x = \sqrt[3]{b \pm 10}$.

Thus $2 = \sqrt[3]{b+10} - \sqrt[3]{b-10}$. Cubing both sides, we get
$8 = b + 10 - 3\sqrt[3]{(b+10)^2(b-10)} + 3\sqrt[3]{(b+10)(b-10)^2} - (b-10)$
$8 = 20 - 3\sqrt[3]{(b^2-100)}\left(\sqrt[3]{b+10} - \sqrt[3]{b-10}\right) = 20 - 6\sqrt[3]{(b^2-100)}$

Then $\sqrt[3]{(b^2-100)} = 2 \Rightarrow b = 6\sqrt{3}$.




18. $BC = 2h_2$ and $S = \frac{1}{2} AD\cdot BC = \frac{1}{2} h_1 \cdot 2h_2 = h_1 h_2$

$AB = 2h_1$ and $S = \frac{1}{2} EC\cdot AB = \frac{1}{2} h_2 \cdot 2h_1 = h_1 h_2$

19. $S = \frac{1+2007}{2}\cdot 1004 = 1004^2 = (1004-4)(1004+4) + 16 = 1008016$.

20. $\frac{a}{b}$ is largest if $a$=1000 and $b$=1002.