6. Let ABC be an isosceles triangle. Suppose that points K and L are chosen on its lateral sides AB and AC respectively so that $AK = CL$ and $\angle ALK + \angle LKB = 60^\circ$. Prove that $KL = BC$.

Solution Let us construct a paralellogramm $BCLM$. The triangles AKL and BMK are equal since $BM = LC = AK$, $BK = AL$, and $\angle KBM = \angle A$ (as alternate angles). Thus for triangle LKM we have $KL = KM$, and $\angle LKM = \angle BKM + \angle LKB = \angle ALK + \angle LKB = 60^\circ$. Therefore the triangle is equilateral and $KL = ML = BC$.

7. Denote by $(a; b)$ the greatest common divisor of a and b. Let n be a positive integer such that $(n; n + 1) < (n; n + 2) < \ldots < (n; n + 35)$.

Prove that $(n; n + 35) < (n; n + 36)$.

Solution

Note that $(n, n + k) = (n, k) \leq k$, that is $(n, n + 1) \leq 1$, $(n, n + 2) \leq 2$, …, $(n, n + 35) \leq 35$.

Thus the equalities above are valid if and only if

$(n, n + 1) = 1$, $(n, n + 2) = 2$, …, $(n, n + 35) = 35$.

Since $(n, n + 4) = 4$ and $(n, n + 9) = 9$, then n is devisable by $4 \cdot 9 = 36$, therefore

$(n, n + 36) = 36 > 35 = (n, n + 35)$.

8. Each of 100 given numbers was increased by 1. Then each number was increased by 1 once more. It is known that the first time the sum of the squares of the numbers was not changed. Find how this sum changed the second time.

Solution Let $a_1, a_2, \ldots, a_{100}$ are given 100 numbers. Since

$$\sum_{i=1}^{100} a_i^2 = \sum_{i=1}^{100} (a_i + 1)^2$$

then we have $\sum_{i=1}^{100} a_i = -50$ and $\sum_{i=1}^{100} (a_i + 2)^2 - \sum_{i=1}^{100} a_i^2 = 4 \sum_{i=1}^{100} a_i + 400 = 200$.

Answer 200

9. Find all integer solutions (x,y) of the following equation

$$(x-2)(x-10) = 3^y$$

Solution Since 3 is a prime number, we have if x-10>0 then $x-2=3^z$ and $x-10=3^t$.

If x-2<0 then $2-x=3^z$ and $10-x=3^t$.

Since $(x-2)-(x-10)=\pm 8$, then y=2 and x=11 or x=1.

Answer Two solutions: (11,2) and (1,2).

10. A 7-digit passcode is called good if all digits are different. A safe box has a good passcode. The safe box opens if a good code is entered and **one** of its digits is equal to the corresponding digit of the passcode. Is there a method of opening the safe box with unknown passcode using less than 7 attempts?

Solution In six attempts, try entering 1234560, 2345610, 3456120, 4561230, 5612340 and 6123450.

Since the correct passcode uses 7 different digits, among first 6 digits of passcode exist a digit from 1 to 6. Since in given 6 passcodes we have digits from 1 to 6 on each place from first up sixth, at least one of its digits is equal to the corresponding digit of the passcode.