1. Let the sum of 2008 integer numbers be odd. To prove that their product is even.

   Is it true inverse, that is if the product of 2008 integers be even then their sum is odd?

Solution. It is evidently that at least one of these numbers is even ,so that their product is even.

The inverse is not true. For instance if for given 2008 integer numbers only two of them be even and all other 2006 integers be odd ,then their sum is even.

2. In how many ways can 2008 be expressed as the sum of one or more consecutive integers.

Solution. Suppose 2008 equals the sum of n consecutive integers starting from k, then the summing up the arithmetic progression we have

$2008=k+(k+1)+\ldots+(k+n-1)= n(2k+n-1)/2$ or $n(2k+n-1)=4016$.

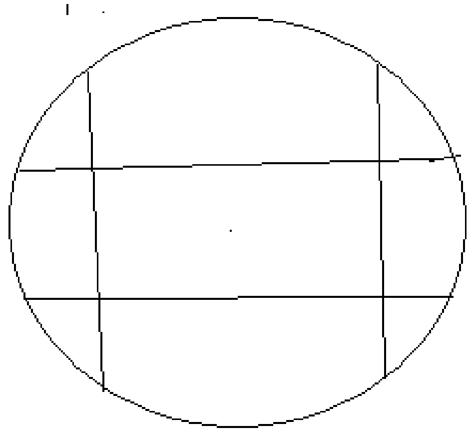
Note that exactly one of n and $(2k+n-1)$ is odd. The only odd factors of 4016 are 1 and 251,therefore either n or $(2k+n-1)$ takes on one of these 2 values. This leaves us 4 possibilities for the ordered pair (n,k).

If $n=1$ then $2k=4016$ and $k=2008$.If $2k+n-1=1$ then $n=4016$ and $k= - 2007$.

If $n=251$ then $k=-117$ .If $2k+n-1=251$ then $n=16$ and $k=236$.

3. Two perpendiculars chords of a circle are at distances a and b respectively from the center. These two chords divide the circle into four pieces. Consider the sum of areas of the largest and the smallest pieces, and the sum of areas of the other two pieces. Find the difference between these two sums.

Solution.



Draw also the mirror images of the chords with respect to the centre. These 4 chords divide the inside of the circle into 9 pieces. The difference between the two sums is the area of the rectangular piece, 4ab.

4. N lamps are controlled by N switches, numbered 1,2,3,…, N. A click on each switch will either turn the lamp ***on*** or ***off***. In the beginning, all the lamps are ***off***. On the first day, all the switches are clicked once. On the second day ,all the switches numbered 2 or a multiple of 2 are clicked once. Similarly on the n^{th} day, all the switches numbered *n* or a multiple of *n* are clicked once, and so on. How many lamps will be ***on*** after the operation on the N^{th} day ,if

a) N=12; b) N=2008 ?

Solution. a) Let N=12, then 3 lamps will be on.

b) Note that the number of clicking on switch with number n is equal to the number of divisors of this number n. For example 7 th switch will be clicked only two times ,and switch with number 12 will be clicked 6 times. Thus after the 2008^{th} operation, only those lamps with number which have an odd number of factors will be ***on***.(It is well known that only perfect squares have odd numbers of factors.) This is equal to the number of perfect squares less than 2008. Since 44^2=1936, and 45^2=2025. Therefore the number of lamps which are ***on*** equal to 44.

5. Do there exist positive integers a,b,c,d such that

$$a/b + c/d = 1,\ a/d + c/b = 2008\ ?$$

Answer: Yes, there are.

Solution: Assume $x = \frac{1}{b}$ and $y = \frac{1}{d}$.Let solve the following system of equations

$$ax + cy = 1$$

$$cx + ay = 2008$$

with respect to unknowns x and y.

We have from first equation

$ax = 1 - cy$ and $x = \frac{1 - cy}{a}$.

Substituting to second one

$$\frac{c(1-cy)}{a} + ay = 2008;\ \ \frac{c - c^2 y}{a} + ay = 2008;\ y(a - \frac{c^2}{a}) = 2008 - \frac{c}{a};$$

$$y = \frac{2008a - c}{a^2 - c^2}\ ;\quad x = \frac{a - 2008c}{a^2 - c^2}\ \text{ and }\ \frac{1}{x} = \frac{a^2 - c^2}{a - 2008c};\ \frac{1}{y} = \frac{a^2 - c^2}{2008a - c}.$$

Here numbers $\frac{1}{x}$ and $\frac{1}{y}$ should be positive integer one. If a=2009 and c=1, then $\frac{1}{x} = 2009^2 - 1$ is the positive integer number. Note that if we increase numbers a and c to k times ,then the numbers $\frac{1}{x}$ and $\frac{1}{y}$ increase also k times so that the equations are satisfied. If put k=2008·2009-1, then $\frac{1}{y}$ be an integer and $\frac{1}{x}$ is integer as before.

Thus the numbers a=2009(2008·2009-1), b=2008·2010(2008·2009-1), c=2008·2009-1 and d=2008·2010 satisfy the conditions of the problem.